# A Kerr Metric Solution in Tetrad Theory of Gravitation

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Using an axial parallel vector field we obtain two exact solutions of a vacuum gravitational field equations. One of the exact solutions gives the Schwarzschild metric while the other gives the Kerr metric. The parallel vector field of the Kerr solution have an axial symmetry. The exact solution of the Kerr metric contains two constants of integration, one being the gravitational mass of the source and the other constant $h$ is related to the angular momentum of the rotating source, when the spin density ${S_{ij}}^{\mu}$ of the gravitational source satisfies $\partial_\mu {S_{ij}}^{\mu}=0$. The singularity of the Kerr solution is studied.

## 1. Introduction

The Riemann Cartan space-time is characterized by the non vanishing of the curvature tensor and the torsion tensor [1]. When we use the teleparallel condition the space-time reduces to the Weitzenböck space-time which is characterized by the torsion tensor only [2]. There are many theories constructed using the Weitzenböck space-time [1, 3, 4]. Among these theories is the theory given by Hayashi and Shirafuji which is called new general relativity and invariant under global Lorentz transformations but not under local Lorentz transformations [1].

This theory contains three dimensionless parameters $a_1$, $a_2$ and $a_3$. Two of these parameters were determined by comparison with solar-system experiments, while an upper bound was estimated for the $a_3$. It was found that the numerical value of $(a_1 + a_2)$ should be very small, consistent with being zero. Throughout this paper we will assume this condition.

Sе́az [5] obtained a set of tetrads satisfying Møller field equations and leading to the Kerr metric by using a particular procedure but without giving an explicit form of this tetrads. Fukui and Hayashi [6] has pointed out that axially symmetric and stationary solution can be obtained in the new general relativity but also without giving an explicit form. Toma [7] gives an exact solution to the vacuum field equation of the new general relativity.

It is our aim to obtain an exact axially symmetric solution of the gravitational field equation of the new general relativity using another procedure different from that used by Toma. In section 2 we briefly review the new general relativity. In section 3 we apply the parallel vector field which is axially symmetric to the field equation of the new general relativity. Two particular exact solutions of the resulting differential equations are obtained. One of these solutions contains one constant of integration and gives the Schwarzschild metric. The other contains two constants of integration and gives the Kerr metric. The physical meaning of these two constants is discussed in section 4. The singularity of the Kerr solution is discussed in section 5. The final section is devoted to the main results.

## 2. The teleparallel theory of gravitation

The fundamental fields of gravitation are the parallel vector fields ${b_k}^{\mu}$. The component of the metric tensor $g_{\mu\nu}$ are related to the dual components ${b^k}_{\mu}$ of the parallel vector fields by the relation

$$g_{\mu\nu} = \eta_{ij} {b^i}_{\mu} {b^j}_{\nu}, \tag{1}$$

where $\eta_{ij} = diag.(-,+,+,+)$. The nonsymmetric connection ${\Gamma^{\lambda}}_{\mu\nu}$* are defined by

$${\Gamma^{\lambda}}_{\mu\nu} = {b_k}^{\lambda} {b^k}_{\mu,\nu}, \tag{2}$$

*Latin indices $(i, j, k, \cdots)$ designate the vector number, which runs from (0) to (3), while Greek indices $(\mu, \nu, \rho, \cdots)$ designate the world-vector components running from 0 to 3. The spatial part of Latin indices is denoted by $(a, b, c, \cdots)$, while that of Greek indices by $(\alpha, \beta, \gamma, \cdots)$.

as a result of the absolute parallelism [1].

The gravitational Lagrangian $L$ of this theory is an invariant constructed from the quadratic terms of the torsion tensor

$$T^{\lambda}{}_{\mu\nu} \stackrel{\text{def.}}{=} \Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\nu\mu}. \tag{3}$$

The following Lagrangian

$$\mathcal{L} \stackrel{\text{def.}}{=} -\frac{1}{3\kappa}\left(t^{\mu\nu\lambda}t_{\mu\nu\lambda} - v^{\mu}v_{\mu}\right) + \zeta a^{\mu}a_{\mu}, \tag{4}$$

is quite favorable experimentally [1]. Here $\zeta$ is a constant parameter, $\kappa$ is the Einstein gravitational constant and $t_{\mu\nu\lambda}, v_{\mu}$ and $a_{\mu}$ are the irreducible components of the torsion tensor:

$$\begin{aligned} t_{\lambda\mu\nu} &= \frac{1}{2}\left(T_{\lambda\mu\nu} + T_{\mu\lambda\nu}\right) + \frac{1}{6}\left(g_{\nu\lambda}V_{\mu} + g_{\mu\nu}V_{\lambda}\right) - \frac{1}{3}g_{\lambda\mu}V_{\nu}, \\ V_{\mu} &= T^{\lambda}{}_{\lambda\nu}, \\ a_{\mu} &= \frac{1}{6}\epsilon_{\mu\nu\rho\sigma}T^{\nu\rho\sigma}, \end{aligned} \tag{5}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is defined by

$$\epsilon_{\mu\nu\rho\sigma} \stackrel{\text{def.}}{=} \sqrt{-g}\delta_{\mu\nu\rho\sigma} \tag{6}$$

with $\delta_{\mu\nu\rho\sigma}$ being completely antisymmetric and normalized as $\delta_{0123} = -1$.

By applying the variational principle to the Lagrangian (4), the gravitational field equation are given by [1]$^{\dagger}$:

$$G_{\mu\nu} + K_{\mu\nu} = -\kappa T_{(\mu\nu)}, \tag{7}$$

$$b^{i}{}_{\mu}b^{j}{}_{\nu}\partial_{\lambda}(\sqrt{-g}J_{ij}{}^{\lambda}) = \lambda\sqrt{-g}T_{[\mu\nu]}, \tag{8}$$

where the Einstein tensor $G_{\mu\nu}$ is defined by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R, \tag{9}$$

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu}\left\{{}^{\rho}_{\sigma\nu}\right\} - \partial_{\nu}\left\{{}^{\rho}_{\sigma\mu}\right\} + \left\{{}^{\rho}_{\tau\mu}\right\}\left\{{}^{\tau}_{\sigma\nu}\right\} - \left\{{}^{\tau}_{\sigma\mu}\right\}\left\{{}^{\rho}_{\tau\nu}\right\}, \tag{10}$$

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}, \tag{11}$$

$$R = g^{\mu\nu}R_{\mu\nu}, \tag{12}$$

and $T_{\mu\nu}$ is the energy-momentum tensor of a source field of the Lagrangian $L_m$

$$T^{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\delta\mathcal{L}_M}{\delta b^{k}{}_{\nu}}b^{k\mu} \tag{13}$$

$^{\dagger}$We will denote the symmetric part by ( ), for example, $A_{(\mu\nu)} = (1/2)(A_{\mu\nu} + A_{\nu\mu})$ and the antisymmetric part by the square bracket [ ], $A_{[\mu\nu]} = (1/2)(A_{\mu\nu} - A_{\nu\mu})$ .

with $L_M = \mathcal{L}_M/\sqrt{-g}$. The tensors $K_{\mu\nu}$ and $J_{ij\mu}$ are defined as

$$K_{\mu\nu} = \frac{\kappa}{\lambda}\left(\frac{1}{2}\left[\epsilon_{\mu}{}^{\rho\sigma\lambda}(T_{\nu\rho\sigma} - T_{\rho\sigma\nu}) + \epsilon_{\nu}{}^{\rho\sigma\lambda}(T_{\mu\rho\sigma} - T_{\rho\sigma\mu})\right] a_\lambda - \frac{3}{2}a_\mu a_\nu - \frac{3}{4}g_{\mu\nu}a^\lambda a_\lambda\right), \tag{14}$$

$$J_{ij\mu} = -\frac{3}{2}b_i{}^\rho b_j{}^\sigma \epsilon_{\rho\sigma\mu\nu}a^\nu, \tag{15}$$

respectively. The dimensionless parameter $\lambda$ is defined by

$$\frac{1}{\lambda} = \frac{4}{9}\zeta + \frac{1}{3\kappa}. \tag{16}$$

In this paper we are going to consider the vacuum gravitational field:

$$T_{(\mu\nu)} = T_{[\mu\nu]} = 0. \tag{17}$$

## 3. Axially symmetric solution

The covariant form of a tetrad space having axial symmetry in spherical polar coordinates, can be written as [5]

$$\left(b^i{}_\mu\right) = \begin{pmatrix} \dfrac{i}{A} & 0 & 0 & iB \\ 0 & C\cos\phi & 0 & -rF\sin\phi \\ 0 & 0 & E & 0 \\ 0 & C\sin\phi & 0 & rF\cos\phi \end{pmatrix}, \tag{18}$$

where $A$, $B$, $C$, $E$ and $F$ are five unknown functions of $(r)$ and $(\theta)$ and he zeroth vector $b^i{}_\mu$ has the factor $i = \sqrt{-1}$ to preserve Lorentz signature. We consider an asymptotically flat space-time in this paper, and impose the boundary condition that for $r \to \infty$ the tetrad (18) approaches the tetrad of Minkowski space-time, $\left(\underset{i}{\lambda}{}_\mu\right) = \mathrm{diag}(i, \delta_a{}^\alpha)$.

Applying (18) to the field equations (7) and (8), we get

$$\frac{-1}{2r^2A^2E^3F^2}\Bigg(B^2E^2{A_r}^2 + 2ABE^3A_rB_r + 2r^2A^2E^2FE_rF_r + 2rA^2E^2F^2E_r - 2r^2AE^2F^2A_rE_r$$
$$+2r^2A^2C^2EFF_{\theta\theta} - 2r^2A^2C^2FE_\theta F_\theta - 2ABC^2EA_\theta B_\theta + 2r^2AC^2F^2A_\theta E_\theta + 4r^2C^2EF^2{A_\theta}^2$$
$$-2r^2AC^2EF^2A_{\theta\theta} - 2rAE^3F^2A_\theta - 2r^2AC^2EFA_\theta F_\theta - 2r^2AE^3FA_rF_r + A^2E^3{B_r}^2$$
$$-B^2C^2E{A_\theta}^2 - A^2C^2E{B_\theta}^2\Bigg) = 0, \tag{19}$$

$$\frac{1}{r^2A^2CEF^2}\Bigg(ABCEA_\theta B_r + B^2CEA_rA_\theta + ABCEA_rB_\theta - 2r^2CEF^2A_rA_\theta - rA^2CEFF_\theta$$
$$-r^2A^2CEFF_{r\theta} + A^2CEB_rB_\theta + r^2A^2EFC_\theta F_r + rA^2EF^2C_\theta + r^2A^2CFE_rF_\theta$$
$$-r^2AEF^2A_rC_\theta - r^2ACF^2A_\theta E_r + r^2ACEF^2A_{r\theta}\Bigg) = 0, \tag{20}$$

$$\frac{2}{r^2A^2C^3F^2}\Bigg(2rACE^2F^2A_r + 2r^2AC^3FA_\theta F_\theta + 2r^2ACE^2FA_rF_r + 2r^2ACE^2F^2A_{rr} + 2AC^2F^2A_\theta C_\theta$$
$$-4r^2CE^2F^2{A_r}^2 + 2r^2A^2E^2FC_rF_r + 2rA^2E^2F^2C_r - 2r^2A^2C^2FC_\theta F_\theta + 2ABCE^2A_rB_r$$
$$-2r^2A^2CE^2FF_{rr} - 4rA^2CE^2FF_r - 2r^2AE^2F^2A_rC_r - 2ABC^3A_\theta B_\theta - A^2C^3{B_\theta}^2$$
$$-B^2C^3{A_\theta}^2 + A^2CE^2{B_r}^2 + B^2CE^2{A_r}^2\Bigg) = 0, \tag{21}$$

$$\frac{-1}{2r^2A^2C^3E^3F^2}\Bigg(3B^4CE^3{A_r}^2 + r^2A^2CE^3F^2{B_r}^2 + r^2A^2C^3EF^2{B_\theta}^2 - 2r^2A^2BE^3F^2B_rC_r$$
$$-2r^2A^2BC^3F^2B_\theta E_\theta + 3A^2B^2C^3E{B_\theta}^2 - 4rA^2BCE^3F^2B_r + 3A^2B^2CE^3{B_r}^2 - 7r^2B^2C^3EF^2{A_\theta}^2$$
$$+2r^2A^2BC^2EF^2B_\theta C_\theta + 2r^2A^2BCE^2F^2B_rE_r - 4r^2A^2BCE^3FB_rF_r + 6AB^3CE^3A_rB_r$$
$$-4r^2AB^2CE^3FA_rF_r - 4r^2A^2BC^3EFB_\theta F_\theta - 4r^2AB^2C^3EFA_\theta F_\theta - 7r^2B^2CE^3F^2{A_r}^2$$
$$+6AB^3C^3EA_\theta B_\theta - 4rAB^2CE^3F^2A_r + 2rA^2BC^2E^3FB_r + 2rAB^2C^2E^3FA_r - 2r^4AC^3EF^4A_{\theta\theta}$$
$$+2r^2A^2BC^3EF^2B_{\theta\theta} + 2r^2A^2BCE^3F^2B_{rr} + 2r^4AC^3F^4A_\theta E_\theta - 2r^2ABC^3EF^2A_\theta B_\theta + 2r^4AE^3F^4A_rC_r$$
$$-2r^2ABCE^3F^2A_rB_r - 2r^4AC^2EF^4A_\theta C_\theta + 2r^2AB^2C^2EF^2A_\theta C_\theta + 2rA^2B^2E^3F^2C_r - 2r^4A^2E^2F^4C_rE_r$$
$$-2r^4A^2C^2F^4C_\theta E_\theta - 2r^4ACE^2F^4A_rE_r + 4r^4C^3EF^4{A_\theta}^2 + 4r^4CE^3F^4{A_r}^2 + 2r^4A^2CE^2F^4E_{rr}$$
$$-2A^2B^2C^3EFF_{\theta\theta} - 2r^2A^2B^2CE^3FF_{rr} - 2r^2A^2B^2CE^2F^2E_{rr} - 2r^2A^2B^2C^2EF^2C_{\theta\theta} + 2r^2AB^2C^3EF^2A_{\theta\theta}$$
$$+2r^2AB^2CE^3F^2A_{rr} + 2r^4A^2C^2EF^4C_{\theta\theta} - 2r^4ACE^3F^4A_{rr} + 2r^2A^2B^2C^3FE_\theta F_\theta + 2r^2A^2B^2C^2F^2C_\theta E_\theta$$
$$-4rA^2B^2CE^3F_r - 2rA^2B^2CE^2F^2E_r + 2r^2A^2B^2E^3FC_rF_r + 2r^2A^2B^2E^2F^2C_rE_r - 2r^2AB^2E^3F^2A_rC_r$$
$$+2r^2AB^2CE^2F^2A_rE_r - 2r^2AB^2C^3F^2A_\theta E_\theta - 2r^2A^2B^2C^2EFC_\theta F_\theta$$
$$-2r^2A^2B^2CE^2FE_rF_r + 3B^4C^3E{A_\theta}^2\Bigg) = 0, \tag{22}$$

$$\frac{1}{2r^2A^3C^3E^3F^2}\Bigg(2r^2A^2BCE^3FF_{rr} + 2r^2A^2BC^2EF^2C_{\theta\theta} + 2r^2A^2BCE^2F^2E_{rr}$$
$$+4r^2BCE^3F^2{A_r}^2 - r^2A^2C^3F^2B_{\theta\theta} + 2r^2A^2BC^3EFF_{\theta\theta} - r^2ABC^3EF^2A_{\theta\theta} - r^2A^2CE^3F^2B_{rr}$$
$$-r^2ABCE^3F^2A_{rr} + r^2A^2E^3F^2B_rC_r + 2rA^2CE^3F^2B_r + r^2ABE^3F^2A_rC_r$$
$$+r^2ABC^3F^2A_\theta E_\theta + 2r^2ACE^3F^2A_rB_r - r^2A^2C^2EF^2B_\theta C_\theta - r^2ABC^2EF^2A_\theta C_\theta$$
$$+r^2A^2C^3F^2B_\theta E_\theta - r^2A^2CE^2F^2B_rE_r - r^2ABCE^2F^2A_rE_r - 6AB^2CE^3A_rB_r$$
$$+2rABCE^3F^2A_r + 2r^2A^2CE^3FB_rF_r + 2r^2ABCE^3FA_rF_r + 2r^2A^2C^3EFB_\theta F_\theta - 6AB^2C^3EA_\theta B_\theta$$
$$+2r^2ABC^3EFA_\theta F_\theta - 3A^2BC^3E{B_\theta}^2 - 3A^2BCE^3{B_r}^2 - 3B^3C^3E{A_\theta}^2 + 2r^2AC^3EF^2A_\theta B_\theta$$
$$+4r^2BC^3EF^2{A_\theta}^2 - 3B^3CE^3{A_r}^2 - 2r^2A^2BC^3FE_\theta F_\theta + 2r^2A^2BCE^2FE_rF_r + 2rA^2BCE^2F^2E_r$$

$$
-2rA^2BE^3F^2C_r - 2r^2A^2BE^3FC_rF_r + 2r^2A^2BC^2EFC_\theta F_\theta - 2r^2A^2BC^2F^2C_\theta E_\theta + 4rA^2BCE^3FF_r
$$
$$
-2r^2A^2BE^2F^2C_rE_r - rA^2C^2E^3FB_r - rABC^2E^3FA_r\Bigg) = 0, \tag{23}
$$

$$
\frac{1}{2r^2A^4C^3E^3F^2}\Bigg(2r^2A^2CE^2F^2E_{rr} - 2r^2A^2E^3FC_rF_r + 4rA^2CE^3FF_r + 2rA^2CE^2F^2E_r
$$
$$
+2r^2A^2CE^2FE_rF_r + 2r^2A^2C^3EFF_{\theta\theta} - 2r^2A^2C^3FE_\theta F_\theta + 2r^2A^2C^2EFC_\theta F_\theta + 2r^2A^2CE^3FF_{rr}
$$
$$
-2r^2A^2C^2F^2C_\theta E_\theta - 2r^2A^2E^2F^2C_rE_r + 2r^2A^2C^2EF^2C_{\theta\theta} - 6ABC^3EA_\theta B_\theta - 2rA^2E^3F^2C_r
$$
$$
-6ABCE^3A_rB_r - 3A^2C^3EB_\theta{}^2 - 3B^2EC^3A_\theta{}^2 - 3A^2CE^3B_r{}^2 - 3B^2CE^3A_r{}^2\Bigg) = 0, \tag{24}
$$

$$
\frac{1}{2rA^3C^3E^3F}\Bigg(2rBC^3EFA_\theta{}^2 - rA^2C^3EFB_{\theta\theta} + rA^2C^3FB_\theta E_\theta + 2rBCE^3FA_r{}^2 - rA^2CE^3FB_{rr}
$$
$$
+rABC^3FA_\theta E_\theta + rA^2E^3FB_rC_r - rABCE^3FA_{rr} + rABE^3FA_rC_r + A^2C^2E^3B_r + ABC^2E^3A_r
$$
$$
-rABC^3EFA_{\theta\theta} - rA^2CE^2FB_rE_r - rABC^2EFA_\theta C_\theta - rABCE^2FA_rE_r
$$
$$
-rA^2C^2EFB_\theta C_\theta\Bigg) = 0, \tag{25}
$$

where $A_r = dA/dr$, $A_{rr} = d^2A/dr^2$, $A_\theta = dA/d\theta$ $A_{\theta\theta} = d^2A/d\theta^2$.

A special vacuum solutions of equations (19)$\sim$(25) are

I) The solution in which the five functions $A$, $B$, $D$, $F$ and $H$ take the form

$$
\begin{aligned}
A &= \frac{1}{\sqrt{1-\frac{a_1}{r}}},\\
B &= 0,\\
C &= \sqrt{\frac{1}{1-\frac{a_1}{r}}},\\
E &= r,\\
F &= \sin\theta,
\end{aligned} \tag{26}
$$

where $a_1$ is a constant of integration. Using (26) in (18), the metric tensor takes the form

$$
ds^2 = -\eta_1 dt^2 + \frac{dr^2}{\eta_1} + r^2 d\Omega^2, \tag{27}
$$

where

$$
\eta_1(r) = (1 - \frac{a_1}{r}). \tag{28}
$$

which is the Schwarzschild solution when the constant $a_1 = 2m$.

II) The solution in which the five functions $A$, $B$, $D$, $F$ and $H$ take the form

$$
\begin{aligned}
A &= \frac{1}{\sqrt{1-\frac{a\rho}{\Sigma}}},\\
B &= \frac{-ah\rho\sin^2\theta}{\sqrt{\Sigma(\rho^2+h^2\cos^2\theta-a\rho)}},\\
C &= \sqrt{\frac{\Sigma}{\Delta}},\\
E &= \sqrt{\Sigma},\\
F &= \sqrt{(\rho^2+h^2)\frac{\sin^2\theta}{\rho^2}+\frac{ah^2\rho\sin^4\theta}{\rho^2\Sigma}+\frac{a^2h^2\sin^4\theta}{\Sigma(\rho^2+h^2\cos^2\theta-a\rho)}},
\end{aligned}
\tag{29}
$$

where $a$, $h$ are two constants of integration and $\Sigma$, $\Delta$ are given by

$$
\Sigma = \rho^2+h^2\cos^2\theta, \qquad \Delta = \rho^2+h^2-a\rho. \tag{30}
$$

The metric tensor associated with the tetrad (18) in this case is given by

$$
ds^2 = -(1-\frac{a\rho}{\Sigma})dt^2+\frac{\Sigma}{\Delta}d\rho^2+\Sigma d\theta^2+\left\{(\rho^2+h^2)\sin^2\theta+\frac{a\rho h^2\sin^4\theta}{\Sigma}\right\}d\phi^2+2\frac{a\rho h\sin^2\theta}{\Sigma}dtd\phi, \tag{31}
$$

which is the Kerr metric written in the Boyer-Lindquist coordinates. Using the solution (29) the tetrad (18) takes the form

$$
\left(b^i{}_\mu\right) = \begin{pmatrix}
i\sqrt{1-\frac{a\rho}{\Sigma}} & 0 & 0 & -i\frac{ah\rho\sin^2\theta}{\sqrt{\Sigma(\rho^2+h^2\cos^2\theta-a\rho)}}\\
0 & \sqrt{\frac{\Sigma}{\Delta}}\cos\phi & 0 & -\rho\sqrt{(\rho^2+h^2)\frac{\sin^2\theta}{\rho^2}+\frac{ah^2\rho\sin^4\theta}{\rho^2\Sigma}+\frac{a^2h^2\sin^4\theta}{\Sigma(\rho^2+h^2\cos^2\theta-a\rho)}}\sin\phi\\
0 & 0 & \sqrt{\Sigma} & 0\\
0 & \sqrt{\frac{\Sigma}{\Delta}}\sin\phi & 0 & \rho\sqrt{(\rho^2+h^2)\frac{\sin^2\theta}{\rho^2}+\frac{ah^2\rho\sin^4\theta}{\rho^2\Sigma}+\frac{a^2h^2\sin^4\theta}{\Sigma(\rho^2+h^2\cos^2\theta-a\rho)}}\cos\phi
\end{pmatrix}. \tag{32}
$$

Thus a vacuum solution which gives the Kerr metric has been given. The parallel vector fields (32) are axially symmetric in the sense that they are form invariant under the transformation

$$
\begin{aligned}
&\bar{\phi}\to\phi+\delta\phi, \qquad \bar{b}^0{}_0\to b^0{}_0, \qquad \bar{b}^1{}_1\to b^1{}_1\cos\delta\phi+b^3{}_3\sin\delta\phi,\\
&\bar{b}^2{}_2\to b^2{}_2, \qquad \bar{b}^3{}_3\to b^3{}_3\cos\delta\phi-b^1{}_1\sin\delta\phi.
\end{aligned}
\tag{33}
$$

The solution (26) satisfy the field equation (7) and (8), but the solution (32) is a solution to the field equation (7) only, i.e., the solution (32) is a solution of general relativity. It is

of interest to note that general relativity has a solution which give the Kerr metric in which the parallel vector fields take the form

$$
\left(b^i{}_\mu\right)_{Sq} = \begin{pmatrix} i\sqrt{1-\dfrac{a\rho}{\Sigma}} & 0 & 0 & -i\dfrac{ah\rho\sin^2\theta}{\sqrt{\Sigma(\rho^2+h^2\cos^2\theta-a\rho)}} \\ 0 & \sqrt{\dfrac{\Sigma}{\Delta}} & 0 & 0 \\ 0 & 0 & \sqrt{\Sigma} & 0 \\ 0 & 0 & 0 & \rho\sqrt{(\rho^2+h^2)\dfrac{\sin^2\theta}{\rho^2}+\dfrac{ah^2\rho\sin^4\theta}{\rho^2\Sigma}+\dfrac{a^2h^2\sin^4\theta}{\Sigma(\rho^2+h^2\cos^2\theta-a\rho)}} \end{pmatrix}, \tag{34}
$$

where $\Sigma$ and $\Delta$ are given by (30). As is clear from (34) that this is just the square root of the Kerr metric.

## 4. The Physical Meaning of $a$ and $h$

Following Toma [7] to clarify the physical meaning of the constants $a$ and $h$ in the solution (32), we consider the weak field approximation [8]

$$
b^k{}_\mu(x) = \delta^k{}_\mu + a^k{}_\mu, \quad |a^k{}_\mu| << 1, \tag{35}
$$

the field $a_{\mu\nu}$ can be expressed as

$$
a_{\mu\nu} = \frac{1}{2}h_{\mu\nu} + A_{\mu\nu}, \tag{36}
$$

with $h_{\mu\nu} = h_{\nu\mu}$ and $A_{\mu\nu} = -A_{\nu\mu}$. The gravitational field equations (7) and (8) takes the forms

$$
\begin{aligned} \Box\overline{h}_{\mu\nu} &= -2\kappa T_{(\mu\nu)}, \\ \Box A_{\mu\nu} &= -\lambda T_{[\mu\nu]}. \end{aligned} \tag{37}
$$

Here the d'Alembertian operator is given by $\Box = \partial^\mu\partial_\mu$, $\overline{h}_{\mu\nu}$ denotes

$$
\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h, \qquad h = \eta_{\mu\nu}h^{\mu\nu}. \tag{38}
$$

Since we have solve the field equations (7) and (8) in the vacuum case then from (37) we require

$$
A_{\mu\nu} = 0. \tag{39}
$$

The tensor $T_{[\mu\nu]}$ is related to the spin density

$$S_{kl}{}^{\mu} = i\frac{\partial \mathcal{L}_m}{\partial \psi_{,\mu}} S_{kl}\psi \tag{40}$$

of the source field $\psi$ through

$$\sqrt{-g}T_{[\mu\nu]} = \frac{1}{2} b^k{}_\mu b^i{}_\nu \partial_\lambda S_{kl}{}^{\lambda}. \tag{41}$$

Here $S_{kl}$ is the representation group of the Lie algebra of the Lorentz group to which $\psi$ belongs. Using (39) in (37) we get

$$\partial_\mu S_{kl}{}^{\mu} = 0. \tag{42}$$

Using the condition (42), the physical meaning of $a$ and $h$ are given by

$$a = \frac{\kappa M}{4\pi}$$
$$h = -\frac{J}{M}, \tag{43}$$

where M is the gravitational mass of a central gravitating body and J represent the angular momentum of the rotating source. The relation (43) is obtained by comparing the metric (31) with the metric [7, 9]

$$ds^2 = -\left(1 - \frac{\kappa M}{4r\pi}\right) dt^2 + \left(1 + \frac{\kappa M}{4r\pi}\right) dx^a dx^a - \frac{\kappa J}{2r\pi} \sin^2\theta dt d\phi. \tag{44}$$

## 5. Study of Singularities

In teleparallel theory of gravity by singularity of space-time we mean the singularity of the scalar concomitants of the torsion and curvature tensors.

The space-time given by (32) does not have singularities for the Ricci scalar and for the concomitants of the Ricci tensor because we have

$$R = R^{\mu\nu} R_{\mu\nu} = 0, \tag{45}$$

using (10) and (32) we obtain

$$R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = -12a^2 \frac{(h^6 \cos\theta^6 - 15\rho^2 h^4 \cos\theta^4 + 15\rho^4 h^2 \cos^2\theta - \rho^6)}{(\rho^2 + h^2 \cos^2\theta)^6}. \tag{46}$$

As is clear from (46) that there is a singularity at $(\rho,\ \theta) = (0,\ \pi/2)$ when $h \neq 0$, which agrees with the singularity of solution (34). The scalar of the torsion tensor and the irreducible components of it [1] are given by

$$
\begin{aligned}
T^{\mu\nu\lambda}T_{\mu\nu\lambda} &= \frac{F_1(\rho,\theta)}{\sin^{3/2}\theta(\rho^2+h^2-a\rho)(\rho^2+h^2\cos^2\theta-a\rho)^{5/2}(\rho^2+h^2\cos^2\theta)^3},\\
t^{\mu\nu\lambda}t_{\mu\nu\lambda} &= \frac{F_2(\rho,\theta)}{\sin^{3/2}\theta(\rho^2+h^2-a\rho)(\rho^2+h^2\cos^2\theta-a\rho)^{5/2}(\rho^2+h^2\cos^2\theta)^3},\\
V^{\mu}V_{\mu} &= \frac{F_3(\rho,\theta)}{\sin^2\theta(\rho^2+h^2-a\rho)(\rho^2+h^2\cos^2\theta)^3},\\
a^{\mu}a_{\mu} &= \frac{F_4(\rho,\theta)}{(\rho^2+h^2-a\rho)^3(\rho^2+h^2\cos^2\theta)^5},
\end{aligned}
\tag{47}
$$

where $F_1(\rho,\theta)$, $F_2(\rho,\theta)$, $F_3(\rho,\theta)$ and $F_4(\rho,\theta)$ are some complicated functions in $\rho$ and $\theta$. From (47) we see that, when $h \neq 0$, there are singularities of the scalar of the torsion tensor, the scalar of the traceless part and the scalar of the basic vector given by $\theta = 0$ and/or $(\rho,\ \theta) = (0,\ \pi/2)$, but for the axial vector the singularity is given by $(\rho,\ \theta) = (0,\ \pi/2)$. Also all the above scalars have a common singularity at $\rho = 0$ when $h = 0$

## 6. Main results

The results of the proceeding sections can be summarized as follow:

1) An exact solution (26) for the field equations (7) and (8) which gives the Schwarzschild metric has been obtained.

2) An exact solution (32) for the field equation (7) which gives the static Kerr metric is obtained. It is axially symmetric, but differs from (34) of general relativity. There is a modification of the Kerr metric for a source that carries an electric charge e. Replacing the definition of $\Delta$ given in (30) by

$$\Delta = \rho^2 + h^2 - a\rho + e^2,$$

leads to the Kerr-Newman metric [10]. It is shown that special relativity for the spin $\frac{1}{2}$ electron can be seen to emerge from Kerr-Newman metric [11]. Also it is shown that [11] a particle can be treated as a relativistic vortex, that is a vortex where the velocity of a circulation equals that of light or a spherical shell, whose constituents are again rotating with the velocity of light or as a black hole described by the Kerr-Newman metric for a spin $\frac{1}{2}$ particles.

3) The space-time given by (32) has a singularity at $(\rho,\ \theta) = (0,\ \pi/2)$ for the concomitants of the Riemann Christoffel tensor and the concomitants of the axial vector part when $h \neq 0$ but for the concomitants of the torsion, tracless part and the basic vector there are singularities

at $(\rho,\ \theta) = (0,\ \pi/2)$ and/or $\theta = 0$ when $h \neq 0$ and all the concomitants have a common singularity at $\rho = 0$ and $h = 0$.

4) There are two solutions of general relativity both of them gave the Kerr metric, but the parallel vector fields reproducing them are quite different in its structures in spite that the singularities of them are quite the same. We believe that the physical contents of these two parallel vector fields (32) and (34) are quite different and this will be our future work by studying the energy contents for the both tetrad.

5) We have solved the non linear partial differential equations (19)$\sim$(25) and obtained two different exact vacuum solutions. From those partial differential equation one can get many physical solutions and may be can solve them in the general case. This also will be our future work.

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